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On some representations of the Drinfel'd double ☆

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Abstract

For H a finite-dimensional semisimple Hopf algebra over an algebraically closed field of characteristic zero the induced representations from H and H^* to the Drinfel'd double $D(H)$ are studied. The product of two such representations is a sum of copies of the regular representation of $D(H)$. The action of certain irreducible central characters of H^* on the simple modules of H is considered. The modules that receive trivial action from each such irreducible central character are precisely the constituents of the tensor powers of the adjoint representation of H.

Keywords: Hopf algebras; Character ring; Drinfel'd double

Introduction

Let H be a finite-dimensional semisimple Hopf algebra over an algebraically closed field of the characteristic zero. The Drinfel'd double of H was introduced by Drinfel'd in order to provide new quasitriangular Hopf algebras. The representation theory of $D(H)$ has been intensively studied in the last years. Kaplansky's tenth conjecture states that the dimension of each simple H-module divides the dimension of H. Since the conjecture was proved for $D(H)$ [2,22] different relations between the category of H-modules and

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that of $D(H)$-modules have been considered. These relations were also used in classifying semisimple Hopf algebras of certain dimensions (see [13] and references there).

In the first section some basic facts about the character ring of H are recalled. Section 2 is concerned with the study of the induction and the restriction functors between H-modules and $D(H)$-modules. The composition of the above two functors is computed. It is shown that $M\uparrow^{D(H)}\downarrow_H \cong \sum_{N\in \mathrm{Irr}(H)} N^* \otimes M \otimes N$.

The trivial module of H induced to $D(H)$ is considered in Section 3. Study of the restriction to H^* of each simple constituent of this module leads to the definition of a set $K(H)$ of irreducible central H^*-characters. The set of the simple constituents of all tensor powers of the adjoint representation of H is characterized as being the set of all simple modules receiving the trivial action from each character in $K(H)$.

When H has a commutative character ring (for example when H is a quasitriangular Hopf algebra) these irreducible central H^*-characters correspond to the central group-like elements of H. Action of these H^*-characters on the set of simple modules of H is considered in Section 4. An application of these results is given in the last section where the Grothendieck ring structure of the Drinfel'd double of the unique nontrivial semisimple eight-dimensional Hopf algebra is described.

For simplicity, the ground field k is assumed to be algebraically closed of characteristic zero even though some of the results also work for characteristic $p > 0$. Algebras and coalgebras are defined over the ground field k; comultiplication, counit and antipode of a Hopf algebra are respectively denoted by Δ, ϵ and S. All the other Hopf algebra notations are those used in [12].

1. The character ring $C(H)$

Let H be a finite-dimensional semisimple Hopf algebra over an algebraically closed field k. Its character ring $C(H)$ is the finite-dimensional k-algebra with basis given by the characters of the irreducible H representations. We denote these characters by χ_i, for $i = 0, \dots, r$ where χ_0 is the trivial character, which is the unit of the k-algebra $C(H)$. Since $C(H)$ is a semisimple k-algebra (see [25]) the Artin–Wedderburn theorem implies that $C(H)$ is a product of matrix rings:

$$C(H) = M_{p_0}(k) \times M_{p_1}(k) \times M_{p_2}(k) \times \cdots \times M_{p_f}(k). \tag{1.1}$$

Since H^* is also semisimple and $t \in C(H)$ being a cocommutative element [21] we may assume that the first block matrix corresponds to the primitive central idempotent $t \in H^*$, the integral of H^* with $t(1) = 1$. Therefore, t generates a one-dimensional two-sided ideal inside $C(H)$ and we have $p_0 = 1$. $C(H)$ admits an associative symmetric nondegenerate bilinear form given by $\langle \chi, \mu \rangle = \chi\mu(\Lambda)$, where Λ is the integral in H with $\epsilon(\Lambda) = 1$. From the orthogonality relations [8], we know that $\{\chi_i, \chi_{i^*}\}$ form dual bases for this bilinear form, where $\chi_{i^*} = S(\chi_i)$. On the other hand, semisimplicity of $C(H)$ implies that $\langle\, , \rangle =$

$\sum_s a_s \mathrm{tr}|_s()$ for some nonzero elements $a_s \in k$ where $\mathrm{tr}|_s()$ represents the trace on the matrix block $M_{p_s}(k)$. In fact, Lorenz's proof of class equation given in [9] shows that

$$a_s = \frac{\dim_k H^* e_s}{p_s \dim_k H}. \tag{1.2}$$

Here e_s are the central primitive idempotents of $C(H)$ for $s = 0, \ldots, f$. If we consider e^s_{uv} to be the matrix entries in $M_{p_s}(k)$ we know that $\{e^s_{uv}, \frac{1}{a_s} e^s_{vu}\}$ are also dual bases on $C(H)$. Therefore, as in [9] we have that

$$\sum_{i=0}^{r} \chi_{i^*} \otimes \chi_i = \sum_{suv} \frac{1}{a_s} e^s_{uv} \otimes e^s_{vu}. \tag{1.3}$$

There is another symmetric nondegenerate bilinear form on $C(H)$, called multiplicity (see [15]), and given by $m(\chi_i, \chi_j) = \delta_{i,j}$ for any two irreducible characters χ_i and χ_j. Thus $\{\chi_i, \chi_i\}$ form dual bases for $m(\,,\,)$. We notice that for any two virtual characters $\chi, \mu \in C(H)$ we have $m(\chi, \mu) = \langle \chi^*, \mu \rangle$. Moreover, for any three characters $\chi, \mu, \eta \in C(H)$ Nichols and Richmond proved in [15] that

$$m(\chi, \mu) = m\big(\chi^*, \mu^*\big) \quad \text{and} \quad m(\chi, \mu\eta) = m\big(\mu^*, \eta\chi^*\big). \tag{1.4}$$

In the next section the following properties of $m(\,,\,)$ are needed:

Proposition 1. *Let H be a semisimple Hopf algebra with the character ring $C(H)$. Then*

(1) $\chi = \sum_{suv} \frac{1}{a_s} m(e^s_{uv}, \chi^*) e^s_{vu}$, *for any* $\chi \in C(H)$.
(2) $m(e^s_{uv}, (e^r_{wt})^*) = \delta_{s,t}\delta_{v,w}\delta_{u,t} a_s$.
(3) $m(e^s_{uv}, e_t) = \delta_{s,t^*}\delta_{u,v} a_s$.

Proof. (1) For any character χ we have $\chi = \sum_{i=0}^r m(\chi_i, \chi)\chi_i = \sum_{i=0}^r m(\chi_{i^*}, \chi^*)\chi_i$. The linear function $m(\cdot, \chi^*)$ applied on the first tensorand of relation (1.3) gives the equality of (1).

(2) Let $\chi = e^r_{wt}$ in (1).

(3) Note that $e_t = \sum_u e^t_{uu}$ and apply (2). □

2. Drinfel'd double $D(H)$

If H is a finite-dimensional Hopf algebra then its Drinfel'd double is a Hopf algebra with underlying vector space $H^* \otimes H$. The coalgebra structure of $D(H)$ is the tensor product coalgebra structure of $H^{*\mathrm{cop}} \otimes H$:

$$\Delta(f \bowtie h) = (f_2 \bowtie h_1) \otimes (f_1 \bowtie h_2) \quad \text{and} \quad \epsilon(f \bowtie h) = f(1)\epsilon(h).$$

The product is defined by the formula:

$$(f \bowtie h)(g \bowtie l) = \langle g_1, S^{-1}h_3 \rangle \langle g_3, h_1 \rangle f g_2 \bowtie h_2 l.$$

The antipode is given by $S(f \bowtie h) = S(h)S^{-1}(f)$.

Since H is semisimple and cosemisimple it follows that $D(H)$ is itself semisimple and cosemisimple [20]. In this case $\Gamma = t \bowtie \Lambda$ is an integral of $D(H)$ satisfying $\epsilon(\Gamma) = 1$ where Λ and t are the idempotent integrals of H respectively H^*. The Hopf algebra H can be canonically considered as a Hopf subalgebra of $D(H)$ via the embedding $h \mapsto \epsilon \bowtie h$. If V is a $D(H)$-module then we get an H-module $V\downarrow_H$, by restricting the $D(H)$ action to H. In this way, we get a map

$$\mathrm{res}_H : C\big(D(H)\big) \to C(H).$$

Now suppose M is an H-module. Since H is embedded in $D(H)$ we may consider the induced module $M\uparrow_H^{D(H)} = D(H) \otimes_H M = H^* \otimes M$. In this way the map

$$\mathrm{ind}_H : C(H) \to C\big(D(H)\big)$$

is defined on the canonical basis of $C(H)$ and then extended by linearity. We use the notation $\chi\uparrow$ for the image of a character $\chi \in C(H)$ under the map ind_H. The following result about the induction functor is needed. Let H be a semisimple Hopf algebra and K a Hopf subalgebra of H. It is known that K is semisimple. If M is an irreducible K-module and e_M is a primitive idempotent of K such that $M \cong K e_M$ then $M\uparrow_K^H \cong H e_M$ [13]. Indeed $M\uparrow_K^H \cong H \otimes_K K e_M \cong H e_M$ since H is free K-module (see [17]).

Proposition 2. *Let H be a finite-dimensional semisimple Hopf algebra and K a Hopf subalgebra of H. Let M be a K-module and V an H-module. Then*

$$V \otimes M\uparrow_K^H \cong (V\downarrow_K \otimes M)\uparrow_K^H.$$

Proof. Frobenius reciprocity and the fact that ind_H and * commutes implies that

$$m_H\big(W, V \otimes M\uparrow_K^H\big) = m_H\big(\big(M\uparrow_K^H\big)^*, W^* \otimes V\big) = m_H\big(\big(M^*\big)\uparrow_K^H, W^* \otimes V\big)$$
$$= m_K\big(M^*, \big(W^* \otimes V\big)\downarrow_K\big),$$

for any H-module W. On the other hand

$$m_H\big(W, (V\downarrow_K \otimes M)\uparrow_K^H\big) = m_K(W\downarrow_K, V\downarrow_K \otimes M) = m_K\big(M^*, W\downarrow_K^* \otimes V\downarrow_K\big)$$
$$= m_K\big(M^*, \big(W^* \otimes V\big)\downarrow_K\big).$$

Therefore $m_H(W, V \otimes M\uparrow_K^H) = m_H(W, (V\downarrow_K \otimes M)\uparrow_K^H)$ for any H-module W which implies that $V \otimes M\uparrow^H \cong (V\downarrow_K \otimes M)\uparrow_K^H$. $\square$

Let $M_0 = k\Lambda$ be the trivial H-module and $A_0 = M_0\uparrow_H^{D(H)}$. The remark before Proposition 2 implies that $A_0 \cong D(H)(\epsilon \bowtie \Lambda) = H^* \bowtie \Lambda$. Then $A_0 \cong H^*$ where $D(H)$ acts on H^* in the following way:

- $\langle h.f, x\rangle = \langle f, S^{-1}h_2xh_1\rangle$ is the left coadjoint action of H on H^*,
- $f.g = fg$ is the left regular H^* action on H^*.

The module A_0 was studied in [26]. It was proved that $\mathrm{End}_{D(H)}(A_0) \cong C(H)$ as algebras inside H^* and the simple $D(H)$-submodules of A_0 are in one to one correspondence with the central primitive idempotents of $C(H)$. With the above notations, it follows that each H^*e_s is a homogeneous component of A_0 and it contains p_s isomorphic copies of a simple $D(H)$-module V_s. (Using the notations of relation (1.1), e_s is the central primitive idempotent of $C(H)$ such that $C(H)e_s \cong M_{p_s}(k)$.) Similarly let B_0 be the module obtained by inducing the trivial module from H^* to $D(H^*)$. Since $D(H) \cong D(H^*)^{\mathrm{cop}}$ as Hopf algebras [20] it follows that B_0 is a $D(H)$-module. Then $B_0 \cong D(H)(t \bowtie 1) = Ht$ where t is an idempotent integral of H^*. Therefore $B_0 \cong H$ where the $D(H)$ action on H is given by:

- $h.l = hl$ is the left regular action of H on H,
- $f.h = \langle f, h_3S^{-1}h_1\rangle h_2$ is the left coadjoint action of H^* on H.

Before studying the relation between A_0 and B_0 we need the following standard fact.

Lemma 3. *Let R be a ring and e and f two idempotents of R. Then* $\mathrm{Hom}_R(Re, Rf) \cong eRf$.

Proposition 4. *Let H be a semisimple Hopf algebra and let A_0 and B_0 be defined as above. Then $A_0 \otimes B_0 \cong D(H)$ and the only common simple $D(H)$ constituent of these two modules is the trivial module.*

Proof. Let

$$\Phi : A_0 \otimes B_0 \to D(H),$$
$$g \otimes a \mapsto (g \leftharpoonup a_1Sa_3) \bowtie a_2.$$

Then Φ is an isomorphism of $D(H)$-modules with the inverse given by

$$\Psi : D(H) \to A_0 \otimes B_0,$$
$$g \bowtie a \mapsto (g \leftharpoonup a_3Sa_1) \otimes a_2.$$

This follows from

$$\Psi\big(\Phi(g \otimes a)\big) = \Psi\big((g \leftharpoonup a_1Sa_3) \bowtie a_2\big) = (g \leftharpoonup a_1Sa_5) \leftharpoonup a_4Sa_2 \bowtie a_3 = g \bowtie a$$

since $S^2 = \mathrm{Id}$ [21] and $A_0 \otimes B_0$ is finite-dimensional. Moreover, Ψ is an isomorphism of $D(H)$-modules since

$$\Psi(g \bowtie a) = (g \leftharpoonup a_3 S a_1) \bowtie a_2 = (g \bowtie a)(\epsilon \otimes 1).$$

Indeed

$$(g \bowtie a)(\epsilon \otimes 1) = g.(\epsilon \otimes a) = g_2 \otimes \langle g_1, a_3 S a_1 \rangle a_2 = (g \leftharpoonup a_3 S a_1) \otimes a_2.$$

Since H is bisemisimple we may consider the idempotent integrals Λ and t of H and H^*, respectively. Then $e = \epsilon \bowtie \Lambda$ and $f = t \bowtie 1$ are idempotents of $D(H)$ and $A_0 \cong D(H)e$, $B_0 \cong D(H)f$. Using previous lemma we have

$$\begin{aligned} m_{D(H)}(A_0, B_0) &= m_{D(H)}\big(D(H)e, D(H)f\big) \\ &= \dim_k \mathrm{Hom}_{D(H)}\big(D(H)e, D(H)f\big) \\ &= \dim_k f D(H)e = 1. \end{aligned}$$

This implies that there is only one common constituent of A_0 and B_0 and this constituent has multiplicity one in both modules. It is easy to see that the trivial $D(H)$-module $k(t \bowtie \Lambda)$ is a constituent for both A_0 and B_0, thus it is the unique common constituent. □

Remark 5. Using Frobenius reciprocity, Proposition 4 implies that the trivial module is the only simple $D(H)$-module whose restriction to both H and H^* contains the trivial module for H and H^*, respectively.

If $\{e_i\}$ is a k-basis in H and $\{f^i\}$ is its dual basis in H^* then the element

$$R = \sum_i (\epsilon \bowtie e_i) \otimes \big(f^i \bowtie 1\big)$$

is an R-matrix, which makes $D(H)$ a quasitriangular Hopf algebra [12]. Therefore $C(D(H))$ is a commutative k-algebra. Let R_{21} be the matrix obtained by interchanging the tensorands of R. The map

$$\begin{aligned} \Phi : D(H)^* &\to D(H), \\ F &\mapsto (\mathrm{id} \otimes F)(R_{21} R) \end{aligned}$$

is bijective showing that the Drinfel'd double is factorizable [14] (see also [22]). Restricted to the character ring of $D(H)$, Φ induces an algebra isomorphism between the character ring and the center of the Drinfel'd double [1]. The image of res_H is $Z(C(H))$ [7]. A different proof of this fact is presented below.

Theorem 6. *Let H be a finite-dimensional semisimple Hopf algebra.*
Then $\mathrm{res}_H : C(D(H)) \to Z(C(H))$ is a surjective algebra map.

Proof. Because the category of modules over $D(H)$ is the center category of the category of H-modules [5] the image of res_H lies in the center of $C(H)$ denoted by $Z(C(H))$. If Λ and t are the nonzero idempotent integrals of H and, respectively, H^*, then $\Gamma = t \otimes \Lambda$ is an idempotent integral of $D(H)$. Let V be an irreducible $D(H)$-module with character μ and let e be the primitive central idempotent of $D(H)$ corresponding to V. According to [11] we have

$$e = \frac{\mu(1)}{n^2}\mu \rightharpoonup \Gamma = \frac{\mu(1)}{n^2}\sum \mu\bigl(S(t_2 \bowtie \Lambda_1)\bigr)t_1 \bowtie \Lambda_2,$$

where $n = \dim_k H$. As pointed out in [7], under the identification of $D(H)^*$ with $H^* \otimes H$, the map Φ restricted to $C(D(H))$ is just the identity map. It follows that

$$E = \Phi^{-1}(e) = \frac{\mu(1)}{n^2}\mu \rightharpoonup \Gamma = \frac{\mu(1)}{n^2}\sum \mu\bigl(S(t_2 \bowtie \Lambda_1)\bigr)t_1 \otimes \Lambda_2$$

is a central primitive idempotent of $C(D(H))$. Therefore

$$\begin{aligned} E{\downarrow_H}(h) = E(\epsilon \bowtie h) &= \frac{\mu(1)}{n^2}\sum \mu\bigl(S(t_2 \otimes \Lambda_1)\bigr)t_1(h)\epsilon(\Lambda_2) \\ &= \frac{\mu(1)}{n^2}\sum \mu\bigl(S\Lambda S t_2 t_1(h)\bigr) = \frac{\mu(1)}{n^2}\epsilon(h)\mu(\epsilon \bowtie \Lambda). \end{aligned}$$

Then $E{\downarrow_H} \neq 0$ if and only if $\mu(\epsilon \bowtie \Lambda) \neq 0$. This is equivalent to $m(\epsilon_H, \mu{\downarrow_H}) \neq 0$. Frobenius reciprocity implies that the simple representation corresponding to μ is a submodule of A_0. Since res_H is an algebra map and A_0 has exactly $\dim_k Z(C(H))$ homogenous components, it follows that $\mathrm{Im}(\mathrm{res}_H) = Z(C(H))$. □

Let $V_0, V_1, \ldots, V_l$ be a complete set of nonisomorphic simple $D(H)$-modules with the characters $\mu_0, \mu_1, \ldots, \mu_l$ and corresponding central primitive idempotents $\xi_0, \xi_1, \ldots, \xi_l$. Assume that V_0 is the trivial $D(H)$-module. Then, as in the previous section write

$$\sum_{i=0}^{l} \mu_{i^*} \otimes \mu_i = \sum_{s=0}^{l} \frac{1}{A_s} E_s \otimes E_s,$$

where E_s are the primitive idempotents of $C(D(H))$ with $\Phi(E_s) = \xi_s$ for $s = 0, \ldots, l$. Notice that $E_0, E_1, \ldots, E_l$ form a linear basis of $C(D(H))$ since $C(D(H))$ is commutative. Without loss of generality suppose that $E_s{\downarrow_H} = e_s$ for $s = 0, \ldots, f$ and $E_s{\downarrow_H} = 0$ for $f < s \leqslant l$. We have the following expression for the composition $\mathrm{res}_H(\mathrm{ind}_H(M)) = M{\uparrow^{D(H)}}{\downarrow_H}$.

Theorem 7. *Let H be a finite-dimensional semisimple Hopf algebra and M be an irreducible representation of H. Then $M{\uparrow^{D(H)}}{\downarrow_H} \cong \sum_{N \in \mathrm{Irr}(H)} N^* \otimes M \otimes N$.*

Proof. Let χ be the irreducible character of M and $\chi_0, \chi_1, \ldots, \chi_r$ be the set of all irreducible characters of H. It is enough to prove that $\chi\uparrow\downarrow = \sum_{i=0}^{r} \chi_{i^*}\chi\chi_i$. Relation (1.3) implies that

$$\sum_{i=0}^{r} \chi_{i^*}\chi\chi_i = \sum_{s=0}^{f} \frac{1}{a_s}\nu_s(\chi)e_s,$$

where $\nu_0, \nu_1, \ldots, \nu_f$ are the characters of $C(H)$ corresponding, respectively, to the central primitive idempotents $e_0, e_1, \ldots, e_f$. Consequently, it is enough to check that

$$\chi\uparrow\downarrow = \sum_{s=0}^{f} \frac{1}{a_s}\nu_s(\chi)e_s,$$

for any character $\chi \in C(H)$. It will be shown that

$$\chi\uparrow = \sum_{s=0}^{f} \frac{1}{a_s}\nu_s(\chi)E_s. \tag{2.1}$$

Then applying res_H the desired equality follows immediately. Therefore it suffices to show that

$$e_{uv}^t\uparrow = \sum_{s=0}^{f} \frac{1}{a_s}\nu_s\big(e_{uv}^t\big)E_s$$

which is equivalent to

$$e_{uv}^t\uparrow = \delta_{u,v}\frac{1}{a_t}E_t.$$

In order to prove this we show first that $e_{uv}^t\uparrow = \delta_{u,v}\frac{a_t}{A_t}E_t$ and then that $A_t = a_t^2$. Frobenius reciprocity and Proposition 1, part (3) implies that

$$m_{D(H)}\big(e_{uv}^t\uparrow, E_s\big) = m_H\big(e_{uv}^t, E_s\downarrow\big) = m_H\big(e_{uv}^t, e_s\big) = \delta_{u,v}\delta_{s,t^*}a_s$$

if $s \leqslant f$ and

$$m_{D(H)}\big(e_{uv}^t\uparrow, E_s\big) = m_H\big(e_{uv}^t, E_s\downarrow\big) = m_H\big(e_{uv}^t, 0\big) = 0$$

if $f < s \leqslant l$. Again Proposition 1, part (1) for $D(H)$ implies that

$$e_{uv}^t\uparrow = \sum_{s=0}^{l} m_{D(H)}\big(E_{s^*}, e_{uv}^t\uparrow\big)\frac{1}{A_s}E_s = \delta_{u,v}\frac{a_t}{A_t}E_t.$$

It follows that $e_t\uparrow = \frac{a_t p_t}{A_t} E_t$ since $e_t = \sum_u e^t_{uu}$ for any $0 \leqslant t \leqslant f$. Thus

$$\chi_0\uparrow = \sum_{s=0}^{f} e_s\uparrow = \sum_{s=0}^{f} \frac{a_s p_s}{A_s} E_s. \tag{2.2}$$

But $A_0\downarrow_H$ is the adjoint representation H_{ad} of H. Therefore $\chi_0\uparrow\downarrow$ is the character χ_{ad} of the adjoint action on H and by [23] $\chi_0\uparrow\downarrow = \sum_{i=0}^{r} \chi_{i^*}\chi_i$. Using again relation (1.3) we get

$$\chi_0\uparrow\downarrow = \sum_{s=0}^{f} \frac{p_s}{a_s} e_s.$$

The above two formulae for $\chi_0\uparrow\downarrow$ give the relation between a_s and A_s, namely $A_s = a_s^2$ for $0 \leqslant s \leqslant f$. □

Theorem 8. *Let M be an H-module and W be an H^*-module. Then*

(1) $M\uparrow_H^{D(H)} \otimes W\uparrow_{H^*}^{D(H)} \cong D(H)^{|M||W|}$,
(2) $m_{D(H)}(M\uparrow_H^{D(H)}, W\uparrow_{H^*}^{D(H)}) = |M||W|$, *where* $|M| = \dim_k M$ *and* $|W| = \dim_k W$.

Proof. (1) Let $1 = \nu_0$ be the character of the trivial representation of H^*. Relation (2.2) applied for H^* instead of H gives that

$$\nu_0\uparrow = \sum_{s=0}^{f'} e'_s\uparrow = \sum_{s=0}^{f'} \frac{a'_s p'_s}{A_s} E'_s, \tag{2.3}$$

where e'_s are the primitive central idempotents of $C(H^*)$ and a'_s, p'_s are the constants of $C(H^*)$ as a_s and p_s were for $C(H)$ in Section 1. E'_s are the primitive idempotents of $C(D(H))$ that have a nonzero restriction to H^*, namely $E'_s\downarrow_{H^*} = e'_s$ (Theorem 6 applied to H^*).

Since $n^2 E_0$ is the regular character of $D(H)$, Proposition 4 implies that $\chi_0\uparrow\nu_0\uparrow = n^2 E_0$ inside $C(D(H))$. Replacement of $\chi_0\uparrow$ and $\nu_0\uparrow$ with the above formulas shows that the only primitive idempotent of $C(D(H))$ with nonzero restrictions to both H and H^* is E_0, the integral of $D(H)^*$. Thus the sets of idempotents $\{E_s \mid 0 \leqslant s \leqslant f\}$ and $\{E'_s \mid 0 \leqslant s \leqslant f'\}$ have only one common element, E_0. This fact together with the formula (2.1) for an induced character given in the proof of the previous theorem implies the equality in part (1).

(2) It is enough to prove the formula in the case when M and W are both simple modules over H and H^*, respectively. Let e_M be a primitive idempotent of H such that $M \cong He_M$ and e_W be a primitive idempotent of H^* such that $W \cong H^* e_W$. Then

$$M\uparrow_H^{D(H)} = D(H) \otimes_H M = H^* \otimes M$$

can be regarded as a submodule of $D(H)$ and

$$M\uparrow_H^{D(H)} = D(H)(\epsilon \bowtie e_M).$$

Similarly

$$W\uparrow_{H^*}^{D(H)} = D(H)(e_W \bowtie 1).$$

Lemma 3 gives

$$\begin{aligned} m_{D(H)}\big(M\uparrow_H^{D(H)}, W\uparrow_{H^*}^{D(H)}\big) &= m_{D(H)}\big(D(H)(\epsilon \bowtie e_M), D(H)(e_W \bowtie 1)\big) \\ &= \dim_k (e_W \bowtie 1)D(H)(\epsilon \bowtie e_M) \\ &= \dim_k\big(e_W H^* \bowtie He_M\big) = |M||W|. \qquad \square \end{aligned}$$

3. The induced trivial representation

In this section we study the restriction of the induced module A_0 to H^*. A_0 is the representation corresponding to $\chi_0\uparrow_H^{D(H)}$. From the description given in previous section, A_0 restricted to H^* is the regular module. The characters of H^*-modules can be viewed as elements of H. Let $d \in H$ be an irreducible H^*-character corresponding to the simple module W_d and let ξ_d be its associated primitive central idempotent of H^*. Since $C(H) \subset H^*$, the H^*-module W_d can be restricted to $C(H)$. Recall that

$$C(H) = k \times M_{p_1}(k) \times M_{p_2}(k) \times \cdots \times M_{p_f}(k).$$

The decomposition of $W_d\downarrow_{C(H)}$ as a direct sum of simple $C(H)$-modules gives a character formula

$$d\downarrow_{C(H)} = \sum_{s=0}^{f} x_s \nu_s,$$

where x_s represents the multiplicity of the corresponding simple $C(H)$-module in $W_d\downarrow_{C(H)}$ and $\nu_0, \ldots, \nu_f$ are the irreducible characters of $C(H)$. Then

$$d\downarrow_{C(H)}\big(e^s_{uv}\big) = x_s \nu_s\big(e^s_{uv}\big) = x_s \delta_{u,v}.$$

On the other hand, $d\downarrow_{C(H)}(e^s_{uv}) = e^s_{uv}(d)$ from the identification of H with H^{**}. It follows that $e^s_{uv}(d) = x_s\delta_{u,v}$ for any matrix entry e^s_{uv}. Recall from the previous section that H^*e_s are the homogenous $D(H)$-components of A_0. Their restriction to H^* is characterized by the following theorem:

Theorem 9. *Let H be a finite-dimensional semisimple Hopf algebra and e_s a central primitive idempotent of the character ring $C(H)$. Then*

$$\big(H^*e_s\big)\downarrow_{H^*} = \sum_{d\in \mathrm{Irr}(H^*)} W_d^{e_s(d)},$$

where the sum is over all irreducible characters d of H^*.

Proof. It is enough to show that $m_{H^*}(W_d, H^*e_s) = e_s(d)$ for every irreducible character d. We denote by π the projection of H^* into the two-sided ideal generated by ξ_d. Then $\pi(e_s) = \sum_{i=0}^{m} f_{ii}$ where f_{ii} are some of the primitive idempotents corresponding to the minimal two-sided ideal $H^*\xi_d$. Therefore, H^*e_s contains m copies of W_d and $m_{H^*}(W_d, H^*e_s) = m$. On the other hand, $e_s(d) = \pi(e_s)(d) = \sum_{i=0}^{m} f_{ii}(d) = m$ since $f_{ii}(d) = 1$ for any $1 \leqslant i \leqslant m$. □

Remark 10. Let χ be a character of H and d a character of H^*. Then

$$\chi^*(d) = d\downarrow_{C(H)}\big(\chi^*\big) = \sum_{s=0}^{r} x_s \nu_s\big(\chi^*\big).$$

Since $\nu_s(\chi^*) = \overline{\nu_s(\chi)}$ (see [16]) it follows that $\chi^*(d) = \overline{\chi(d)}$.

For every irreducible character d of H^* we define $F_d = \sum_{i=0}^{r} \chi_{i^*}(d)\chi_i$. Note that $F_d \in Z(C(H))$ since by relation (1.3) it follows that

$$F_d = \sum_{s=0}^{f} \frac{1}{a_s} e_{uv}^s(d) e_{vu}^s = \sum_{s=0}^{f} \frac{1}{p_s a_s} e_s(d) e_s. \tag{3.1}$$

If g is a group like element of H we have the following characterization for F_g:

Lemma 11. *Suppose H is a finite-dimensional semisimple Hopf algebra and $e_s \in C(H)$ is a primitive central idempotent. Then $\frac{p_s}{a_s} \leqslant n$ and we have equality if and only if $e_s = \xi_d$, the central idempotent associated to an irreducible H^*-character $d \in Z(H)$. In this case $\frac{\epsilon(d)}{n} F_d = \xi_d$.*

Proof. Formula (1.2) for a_s gives

$$\frac{p_s}{a_s} = \frac{np_s^2}{\dim_k H^*e_s} = n\frac{\dim_k C(H)e_s}{\dim_k H^*e_s} \leqslant n.$$

Therefore $\frac{p_s}{a_s} = n$ if and only if $p_s^2 = \dim_k H^*e_s$. It follows that $\dim_k V_s = p_s$ which means that H^*e_s is a homogenous $D(H)$ module. But A_0 is isomorphic to H^* as H^*-modules and $V_s\downarrow_{H^*}$ is a homogenous H^*-module. The proof of Theorem 9 implies that $e_s = \xi_d$ for some irreducible H^* character d. We claim that $d \in Z(H)$. Indeed $d = \frac{n}{\epsilon(d)}(\xi_d \rightharpoonup \Lambda)$ and the map "$\rightharpoonup$" sends $C(H)$ into $Z(H)$. With the above notations

$$x_s = \epsilon(d) \quad \text{and} \quad a_s = \frac{\dim_k H^* e_s}{n p_s} = \frac{\epsilon^2(d)}{n\epsilon(d)} = \frac{\epsilon(d)}{n}.$$

It follows that

$$F_d = \sum_{tuv} \frac{1}{a_t} e^t_{uv}(d) e^t_{vu} = \frac{x_s}{a_s} e_s = \frac{n}{\epsilon(d)} e_s. \qquad \square$$

Let $K(H)$ be the set of all irreducible characters $d \in C(H^*)$ that have the property of the previous lemma. Consequently

$$K(H) = \bigl\{ d \in C(H^*) \mid \xi_d \text{ is a primitive central idempotent of } C(H) \bigr\}.$$

Proposition 12. *Let H be a finite-dimensional semisimple Hopf algebra. An irreducible character $d \in C(H^*)$ acts as $\epsilon(d)\mathrm{Id}$ on H_{ad} if and only if $d \in K(H)$.*

Proof. Suppose $d \in K(H)$. By [23]

$$\chi_{\mathrm{ad}} = \sum_{i=0}^{r} \chi_i \chi_{i^*} = \sum_{s=0}^{r} \frac{p_s}{a_s} e_s \quad \text{and} \quad \chi_{\mathrm{ad}}(d) = \sum_{s=0}^{f} \frac{p_s}{a_s} e_s(d).$$

Since $d \in K(H)$ there is only one central idempotent e_s with $e_s(d) \neq 0$, namely ξ_d. Therefore, $\chi_{\mathrm{ad}}(d) = \epsilon(d)\chi(1)$ by Lemma 11. It follows that d acts as $\epsilon(d)\mathrm{Id}_M$ on each irreducible constituent of H_{ad}. Conversely suppose that d acts as $\epsilon(d)\mathrm{Id}$ on H_{ad}. Then $\chi_{\mathrm{ad}}(d) = n\epsilon(d)$. Formula (1.2) implies that

$$\sum_{s=0}^{f} \frac{n p_s^2}{\dim_k H^* e_s} e_s(d) = n\epsilon(d) \quad \text{or} \quad \sum_{s=0}^{f} \frac{p_s^2}{\dim_k H^* e_s} e_s(d) = \epsilon(d).$$

Since $\dim_k C(H)e_s = p_s^2$ the last relation becomes

$$\sum_{s=0}^{f} \frac{\dim_k C(H)e_s}{\dim_k H^* e_s} e_s(d) = \epsilon(d).$$

The value $e_s(d)$ is a nonnegative integer since it represents the multiplicity of W_d in $H^* e_s$. On the other hand

$$\sum_{s=0}^{f} e_s(d) = \epsilon(d) \quad \text{and} \quad \frac{\dim_k C(H)e_s}{\dim_k H^* e_s} \leqslant 1 \quad \text{for any } 0 \leqslant s \leqslant f.$$

It follows that $\dim_k C(H)e_s = \dim_k H^*e_s$ whenever $e_s(d) \neq 0$. There is only one s with this property since in this case $\frac{p_s}{a_s} = n$ and Lemma 11 implies that $d \in K(H)$. □

Recall that the exponent of a semisimple Hopf algebra is the smallest positive number $m > 0$ such that $h^{[m]} = \epsilon(h)1$ for all $h \in H$. The generalized power $h^{[m]}$ is defined by $h^{[m]} = \sum_{(h)} h_1 h_2 \dots h_m$. The exponent of a finite-dimensional semisimple Hopf algebra is always finite and divides the cube power of dimension of H [3].

Proposition 13. *Let $d \in C(H^*)$ be an irreducible H^* character and $\chi \in C(H)$ be an irreducible H character. Then $|\chi(d)| \leqslant \chi(1)\epsilon(d)$ with equality if and only if d acts as $\alpha \mathrm{Id}_M$ on the irreducible H representation M corresponding to the character χ where α is a root of $\epsilon(d)$.*

Proof. Let W be the irreducible representation of H^* corresponding to the character d. Then W is a right H-comodule. Define the map

$$T : M \otimes W \to M \otimes W,$$
$$m \otimes w \mapsto w_2 m \otimes w_1.$$

If $l = \exp(H)$ then $T^l = \mathrm{Id}_{M\otimes W}$. Therefore T is a semisimple operator and all its eigenvalues are root of unity. It follows that $\mathrm{tr}(T)$ is the sum of all these eigenvalues and in consequence $|\mathrm{tr}(T)| \leqslant \dim_k(M \otimes W) = \chi(1)\epsilon(d)$. It is easy to see that $\mathrm{tr}(T) = \chi(d)$. Indeed if $W_d = \langle x_{1i} \rangle$ is considered as the subspace generated by the first row of co-matrix entries then

$$T(m \otimes x_{1i}) = \sum_{j=0}^{\epsilon(d)} x_{ji} m \otimes x_{1j}$$

which shows that

$$\mathrm{tr}(T) = \sum_{i=0}^{\epsilon(d)} \chi(x_{ii}) = \chi(d).$$

Equality holds if and only if $T = \alpha\chi(1)\epsilon(d)\mathrm{Id}_{M\otimes W}$ for some α root of unity. The above expression for T implies that in this case $x_{ij} m = \delta_{i,j} \alpha m$ for any $1 \leqslant i, j \leqslant \epsilon(d)$. In particular $dm = \alpha\epsilon(d)m$ for any $m \in M$ which shows that d acts as a scalar multiple on M and that scalar is a root of $\epsilon(d)$. The converse is immediate. □

The sets of H^*-characters closed under product and taking $*$ are in bijective correspondence with the Hopf subalgebras of H (see [17]). Let $\langle X \rangle$ denote the Hopf subalgebra of H corresponding to a such set X of characters.

Proposition 14. *Let H be a finite-dimensional semisimple Hopf algebra. Then the set $K(H)$ is closed under multiplicity and *. It generates a Hopf subalgebra $\langle K(H)\rangle$ of H which is the biggest central Hopf subalgebra of H.*

Proof. If $d \in K(H)$ then $d^* \in K(H)$ since $\chi_{\mathrm{ad}}(d^*) = \overline{\chi_{\mathrm{ad}}(d)} = \epsilon(d)n$ and d^* acts with the same scalar on H_{ad}. If $d, d' \in K(H)$ write $dd' = \sum_{i=1}^{q} m_i d_i$ where d_i are irreducible characters of H^*. Then

$$\chi_{\mathrm{ad}}(dd') = \sum_{i=1}^{q} m_i \chi_{\mathrm{ad}}(d_i) \quad \text{and}$$

$$\left|\chi_{\mathrm{ad}}(dd')\right| \leqslant \sum_{i=1}^{q} m_i \left|\chi_{\mathrm{ad}}(d_i)\right| \leqslant \chi_{\mathrm{ad}}(1) \sum_{i=1}^{q} m_i \epsilon(d_i) = \chi_{\mathrm{ad}}(1)\epsilon(dd').$$

It follows by Proposition 13 that $|\chi_{\mathrm{ad}}(d_i)| = \chi_{\mathrm{ad}}(1)\epsilon(d_i)$ for all $i = 0, \ldots, q$ and therefore each d_i acts as a scalar multiple on H_{ad}. Since d_i acts as $\epsilon(d_i)$ multiple of identity on $k\Lambda$ it follows that d_i acts as the same multiple of the identity on each constituent of H_{ad} and therefore $d_i \in K(H)$. If $L \subset Z(H)$ is a Hopf subalgebra of H then L acts as epsilon identity on H_{ad}. It follows that all the irreducible characters of L^* are contained in $K(H)$ and therefore L is contained in $\langle K(H)\rangle$. □

To proceed further, we need to recall the notion of index of a character introduced in [7]. Let H be a semisimple Hopf algebra and V an H module with the corresponding character χ. If $J = \bigcap_{m \geqslant 0} \mathrm{Ann}(V^{\otimes m})$ is the intersection of the annihilators of all the tensor powers of V then J is the largest Hopf ideal contained in the annihilator of V (see [19]). Let A be the matrix of the linear operator L_χ of $C(H)$ corresponding to the standard basis of $C(H)$ given by the irreducible characters of H. Then A has nonnegative integer entries. Following [7], $J = 0$ if and only if A is an indecomposable matrix. In this context, an $(m \times m)$-matrix A is called indecomposable if it is not possible to find a partition of $\{1, 2, \ldots, m\}$ into two sets M and N such that $a_{ij} = 0$ for all $i \in M$ and $j \in N$. The index of imprimitivity of A is the number of eigenvalues of A with the possible greatest absolute value (see [4]). The index of the character χ is defined to be the index of imprimitivity of the matrix A. Recall that the greatest absolute eigenvalue is $\chi(1)$ [7]. In [7] it was proved that if a simple module M is a constituent of two tensor powers $V^{\otimes m}$ and $V^{\otimes l}$ of V then $m - l$ is divisible by the index of χ.

Remark 15. With the above notations, let $K = H/J$ be the quotient Hopf algebra of H. The set of all irreducible modules of K is the set of irreducible constituents of the tensor powers of V. Then $C(K)$ is a subring of $C(H)$ and every primitive idempotent of $C(K)$ can be written as a sum of primitive idempotents of $C(H)$. If χ is central in $C(H)$ then χ is also central in $C(K)$. The corresponding eigenvectors of the linear operator L_χ of $C(H)$ are exactly the primitive idempotents of $C(H)$. Suppose e is a primitive idempotent of $C(K)$ and the corresponding eigenvalue of L_χ restricted to $C(K)$ at e is λ. It follows that all the primitive idempotents of $C(H)$ entering in the decomposition of e give the same eigenvalue λ

for L_χ. In particular, if $e \in C(K)$ is an eigenvector of L_χ corresponding to the eigenvalue λ with a one-dimensional eigenspace on $C(K)$, then e is the sum of all the primitive idempotents of $C(H)$ which are eigenvectors of L_χ with the corresponding eigenvalue λ.

The trivial representation is a constituent of H_{ad} and therefore all the constituents of $H_{\mathrm{ad}}^{\otimes m}$ are also constituents of $H_{\mathrm{ad}}^{\otimes l}$ for any $m < l$ natural numbers. Thus there is a smallest number p such that all the simple constituents of $H_{\mathrm{ad}}^{\otimes (p+q)}$ are the same as those of $H_{\mathrm{ad}}^{\otimes p}$ for any $q \geqslant 0$.

Theorem 16. *Let H be a finite-dimensional semisimple Hopf algebra. An irreducible representation M of H is a constituent of $H_{\mathrm{ad}}^{\otimes p}$ if and only if every central irreducible character $d \in K(H)$ acts as $\epsilon(d)\mathrm{Id}_M$ on M.*

Proof. By the previous proposition every irreducible central character $d \in K(H)$ acts as $\epsilon(d)\mathrm{Id}$ on H_{ad}. The set of all constituents of $H_{\mathrm{ad}}^{\otimes p}$ is closed under multiplication and *. It corresponds to a quotient Hopf algebra $K = H/J$. Then H_{ad} is a module over K and its tensor powers generate all the representations of K. The index of H_{ad} is one since the trivial representation of K appears as a constituent of any power of H_{ad}. In consequence, by [7, Theorem 5.3], the eigenspace of $L_{\chi_{\mathrm{ad}}}$ corresponding to the eigenvalue $n = \chi_{\mathrm{ad}}(1)$ is one-dimensional and it is generated by the idempotent integral t_K of K. Using [8] we have

$$t_K = \frac{1}{\dim_k K} \sum_{\chi \in \mathrm{Irr}(K)} \chi(1)\chi = \frac{1}{\dim_k K} \sum_{\chi, m(\chi, \chi_{\mathrm{ad}}^p) > 0} \chi(1)\chi.$$

As in Proposition 13 $\chi_{\mathrm{ad}} = \sum_{i=0}^{r} \chi_i \chi_{i^*} = \sum_{s=0}^{f} \frac{p_s}{a_s} e_s$ and χ_{ad} is a central element of $C(H)$. Consider the decomposition of t_K as sum of primitive idempotents of $C(H)$. If e_{uu}^s appears in the decomposition of t_K the above remark implies that the eigenvalue of χ_{ad} at e_{uu}^s is n. The above formula of χ_{ad} gives that $\frac{p_s}{a_s} = n$. According to Lemma 11 this implies that $e_s = \xi_d$ with $d \in K(H)$. Thus $t_K = \sum_{d \in K(H)} \xi_d$. From the same Lemma 11 we know that

$$\xi_d = \frac{\epsilon(d)}{n} F_d = \frac{\epsilon(d)}{n} \sum_{i=0}^{r} \chi_{i^*}(d)\chi_i$$

and

$$t_K = \sum_{d \in K(H)} \frac{\epsilon(d)}{n} \sum_{i=0}^{r} \chi_{i^*}(d)\chi_i = \frac{1}{n} \sum_{i=0}^{r} \left\langle \chi_{i^*}, \sum_{d \in K(H)} d\epsilon(d) \right\rangle \chi_i.$$

The two formulas for t_K show that χ is a constituent of χ_{ad}^p if and only if

$$\left\langle \chi, \sum_{d \in K(H)} d\epsilon(d) \right\rangle \neq 0.$$

Equalizing the coefficient of $\chi_0 = \epsilon$ in these two formulas we get that

$$\frac{n}{\dim_k K} = \sum_{d \in K(H)} \epsilon(d)^2. \tag{3.2}$$

Therefore

$$\left\langle \chi, \sum_{d \in K(H)} d\epsilon(d) \right\rangle = \chi(1) \sum_{d \in K(H)} \epsilon(d)^2$$

for every constituent of χ_{ad}^p, otherwise this evaluation is 0. Since $|\chi(d)| \leqslant \chi(1)\epsilon(d)$ we deduce that χ is a constituent of χ_{ad}^p if and only if $\chi(d) = \chi(1)\epsilon(d)$ for every $d \in K(H)$. Therefore M is a constituent of $H_{\mathrm{ad}}^{\otimes p}$ if and only if every central irreducible character $d \in K(H)$ acts as $\epsilon(d)\mathrm{Id}_M$ on M. □

Remark 17. Formula (3.2) gives that $\frac{n}{|K|} = \dim_k \langle K(H) \rangle$ and $M_\chi \in \mathrm{Irr}(H_{\mathrm{ad}}^{\otimes p})$ if and only if $\chi \downarrow_{K(H)} = \chi(1)\epsilon$.

Proposition 18. *Let H be a finite-dimensional semisimple Hopf algebra. Then*

$$\mathrm{Ann}\big(H_{\mathrm{ad}}^{\otimes p}\big) = \omega\big(\big\langle K(H) \big\rangle\big) H,$$

the augmentation ideal of $\langle K(H) \rangle$ extended to H.

Proof. Let J be the annihilator of $H_{\mathrm{ad}}^{\otimes p}$. By the previous theorem every $d \in K(H)$ acts as identity on $H_{\mathrm{ad}}^{\otimes p}$. Therefore $d - \epsilon(d)1$ is in the annihilator of $H_{\mathrm{ad}}^{\otimes p}$. It follows that $\omega(\langle K(H) \rangle)H$ is contained in J. Since

$$\dim_k(H/J) = \dim_k(K) = \frac{n}{\dim_k \langle K(H) \rangle} = \dim_k\big(H/\omega\big(\big\langle K(H) \big\rangle H\big)\big)$$

we conclude that $J = \omega(\langle K(H) \rangle)H$. □

4. An equivalence relation on the set of irreducible characters

If H has a commutative character ring (for example, if H is quasitriangular), then the action of the central characters $d \in K(H)$ on the irreducible representations of H can be described in terms of the restriction functor from $D(H)$-mod to H-mod. In order to establish a relation between this action and the restriction to H of the $D(H)$-modules, a binary relation on the set of irreducible characters of H is introduced. Let χ and μ be two irreducible characters of H corresponding to the irreducible representations M respectively N. We define $\chi \sim \mu$ if there is a simple $D(H)$-module V such that M and N are constituents of $V \downarrow_H$. It is clear that $\sim$ is reflexive and symmetric. Let us remark that this is an equivalence relation in the case when H is the dual of a group algebra kG. Indeed in this case $D(kG^*)$ is isomorphic with $D(kG)$ and the irreducible representations of the latter are described in [24]. They are isomorphic with the induced modules $kG \otimes_{Z_i} M$ where Z_i runs over the centralizers of a set of conjugacy class representatives and M over all the irreducible representations of Z_i. It is easy to see that in this case the relation defined before coincides with the conjugacy relation in the group G which clearly is an equivalence relation. We will see that $\sim$ is not an equivalence relation in general. (See Example 1.) If we consider the transitive closure of this relation: $\chi \approx \mu$ if there are irreducible characters $\chi_1, \chi_2, \ldots, \chi_s$ such that $\chi \sim \chi_1 \sim \chi_2 \sim \cdots \sim \chi_s \sim \mu$, clearly $\approx$ is an equivalence relation.

A description of the equivalence classes of $\approx$ will be given in this section. If $C(H)$ is commutative it will be shown that $\chi \approx \mu$ if and only if they receive the same action from each character $d \in K(H)$. A necessary and sufficient condition for $\sim$ to be an equivalence relation is described in this case. Frobenius reciprocity implies that $\chi \sim \mu$ is equivalent with the fact that V is a constituent of both $M\uparrow$ and $N\uparrow$. Therefore $\chi \sim \mu$ if and only if $m_{D(H)}(M\uparrow, N\uparrow) > 0$ or $m_H(M\uparrow\downarrow, N) > 0$.

Proposition 19. *Let H be a finite-dimensional semisimple Hopf algebra and $\chi_0, \chi_1, \ldots, \chi_r$ the set of all irreducible characters of H. Then for any two characters χ_u and χ_v we have $\chi_u \sim \chi_v$ if and only if there are i and j such that $m(\chi_u, \chi_i\chi_j) > 0$ and $m(\chi_v, \chi_j\chi_i) > 0$.*

Proof. The above remark gives that $\chi_u \sim \chi_v$ if and only if $m(\chi_v, \chi_u\uparrow\downarrow) > 0$. It is enough to prove that $\chi_u\uparrow\downarrow = \sum_{i,j=0}^{r} m(\chi_u, \chi_i\chi_j)\chi_j\chi_i$. Using Theorem 7 this is equivalent with

$$\sum_{i=0}^{r} \chi_{i^*}\chi\chi_i = \sum_{i,j=0}^{r} m(\chi, \chi_i\chi_j)\chi_j\chi_i$$

for any character $\chi \in C(H)$.

The second property of $m(\,,\,)$ given in (1.4) implies

$$\chi_{i^*}\chi = \sum_{j=0}^{r} m(\chi_j, \chi_{i^*}\chi)\chi_j = \sum_{j=0}^{r} m(\chi, \chi_i\chi_j)\chi_j.$$

If we multiply to the right with χ_i and add over i we get the desired equality. □

Remark 20.

(1) If $H = kG^*$ then two kG^*-characters $g, h \in G$ are conjugate if and only if $g = ab$ and $h = ba$ for some $a, b \in G$.
(2) $\chi \sim \chi_0$ if and only if $m(\chi, \chi_{\mathrm{ad}}) > 0$ since $\chi_{\mathrm{ad}} = \chi_0\uparrow\downarrow$.
(3) $m(\chi, \chi_{\mathrm{ad}}) = \mathrm{tr}(L_\chi)$ where L_χ is the linear operator of $C(H)$ given by left multiplication with χ.
Indeed,

$$\mathrm{tr}(L_\chi) = \sum_{i=0}^{r} m(\chi_i, \chi\chi_i) = \sum_{i=0}^{r} m(\chi, \chi_{i^*}\chi_i) = m(\chi, \chi_{\mathrm{ad}}).$$

For the rest of this section we assume that $C(H)$ is a commutative k-algebra (e.g., H-quasitriangular).

In this case the formula from Theorem 7 becomes $\chi\uparrow\downarrow = \chi\chi_{\mathrm{ad}}$. For any two irreducible characters χ and μ we have $\chi \sim \mu$ if and only if $\mathrm{tr}(L_{\chi\mu^*}) > 0$ which is the same as $m(\chi\mu^*, \chi_{\mathrm{ad}}) > 0$. Indeed, from the above remark it follows that $m(\chi, \mu\uparrow\downarrow) = m(\chi, \mu\chi_{\mathrm{ad}}) = m(\chi^*\mu, \chi_{\mathrm{ad}}) = m(\chi\mu^*, \chi_{\mathrm{ad}})$.

Remark 21. Since $C(H)$ is commutative it follows that $p_s = 1$ for any $0 \leqslant s \leqslant f$. Then $\frac{p_s}{a_s} = n$ if and only if $a_s = n$ which is equivalent with $\dim_k H^*e_s = 1$. From the proof of Lemma 11 we deduce that

$$e_s = \frac{1}{n}F_g$$

for some central grouplike element g. Therefore in this case $K(H) = \bar{G}(H)$ where $\bar{G}(H)$ is the group of central grouplike elements of H.

Lemma 22. *Let H be a finite-dimensional semisimple Hopf algebra and assume that $C(H)$ is a commutative k-algebra. Then $\sim$ is an equivalence relation if and only if $H_{\mathrm{ad}}^{\otimes 2}$ and H_{ad} have the same simple constituents.*

Proof. If $\chi \in C(H)$ is a constituent of H_{ad} then all the constituents of $\chi\mu$ are in relation with μ. Indeed, let ν be a constituent of $\chi\mu$. Then $m(\nu, \chi\mu) = m(\chi^*, \mu\nu^*) = m(\chi, \nu\mu^*)$ and $\mathrm{tr}(L_{\nu\mu^*}) \geqslant \mathrm{tr}(L_\chi) > 0$. Therefore, $\nu \sim \mu$.

Assume that $\sim$ is an equivalence relation. Let χ and μ be two simple constituents of H_{ad}. By the previous statement all the constituents of $\chi\mu$ are in relation with both χ and μ. Since χ is in relation with ϵ from transitivity it follows that all the constituents of $\chi\mu$ are in relation with ϵ and therefore they are constituents of H_{ad}.

Suppose that $H_{\mathrm{ad}}^{\otimes 2}$ and H_{ad} have the same simple constituents. Let $\chi \sim \mu$ and $\mu \sim \nu$. We have to prove $\chi \sim \nu$. First relation is equivalent with $m(\mu, \chi\uparrow\downarrow) > 0$ and the second one with $m(\mu, \nu\uparrow\downarrow) > 0$. It follows that $m(\chi\uparrow\downarrow, \nu\uparrow\downarrow) > 0$ which is the same with $m(\chi\chi_{\mathrm{ad}}, \mu\chi_{\mathrm{ad}}) > 0$. Then $m(\chi\mu^*, \chi_{\mathrm{ad}}^2) > 0$. Since $H_{\mathrm{ad}}^{\otimes 2}$ and H_{ad} have the same simple constituents the assertion follows from Remark 20(3) above. □

Remark 23. Assume that $C(H)$ is commutative. Then:

(1) According to Remark 21 we have that $K(H) = \bar{G}(H)$, the group of central group like elements of H. In this case an irreducible representation M of H is a constituent of $H_{\mathrm{ad}}^{\otimes p}$ if and only if every central grouplike element acts as identity on M. Recall form above that p is the smallest number such that all the simple constituents of $H_{\mathrm{ad}}^{\otimes (p+q)}$ are the same as those of $H_{\mathrm{ad}}^{\otimes p}$ for any $q \geqslant 0$.
(2) Let M be a simple module of H. All the other simple modules of H receiving the same action as M from each central grouplike element $g \in \bar{G}(H)$ are exactly the simple constituents of $M \otimes H_{\mathrm{ad}}^{\otimes p}$. Indeed, since $g \in \bar{G}(H)$ acts as identity on $H_{\mathrm{ad}}^{\otimes p}$ it acts via the same scalars on both M and $M \otimes H_{\mathrm{ad}}^{\otimes p}$ and thus on each constituent of the latter. Con-

versely, if g acts the same on M and N then g acts as identity on $M \otimes N^*$. Therefore all the constituents of $M \otimes N^*$ are in $H_{\mathrm{ad}}^{\otimes p}$ which implies that N is a constituent of $M \otimes H_{\mathrm{ad}}^{\otimes p}$.

(3) The two formulas for t_K from the above proof give that $|\bar{G}(H)| = \frac{n}{|K|}$. Thus, $\bar{G}(H)$ is trivial if and only if all irreducible modules of H are constituents of $H_{\mathrm{ad}}^{\otimes p}$.

Corollary 24. *Assume H is a semisimple Hopf algebra with $C(H)$ commutative. Let χ and μ be two irreducible characters of H. Then*

(1) *$\chi \approx \mu$ if and only if $m(\chi, \mu\chi_{\mathrm{ad}}^p) > 0$.*
(2) *$\chi \approx \mu$ if and only if $\frac{1}{\chi(1)}\chi\downarrow_{\bar{G}(H)} = \frac{1}{\mu(1)}\mu\downarrow_{\bar{G}(H)}$.*
(3) *The number of equivalence classes of $\approx$ is equal with the order of $\bar{G}(H)$.*

Proof. (1) Let T be the linear operator of $C(H)$ defined as $\mathrm{res}_H \circ \mathrm{ind}_H$. By Theorem 7 we know that $T(\chi) = \sum_{i=0}^{r} \chi_{i^*}\chi\chi_i = \chi\chi_{\mathrm{ad}}$ for any character $\chi \in C(H)$. Since $\chi \sim \mu$ if and only if $m(\chi, T(\mu)) > 0$ it follows that $\chi \approx \mu$ if and only if $m(\chi, T^m(\mu)) > 0$ for some positive integer m. But $T^m(\mu) = \mu\chi_{\mathrm{ad}}^m$ and (1) follows.

(2) Any irreducible character χ has the property that

$$\chi\downarrow_{k\bar{G}(H)} = \chi(1)\psi$$

where ψ is an irreducible character of $\bar{G}(H)$. Indeed, since any $g \in \bar{G}(H)$ acts as a scalar multiple of identity on the associated representation M_χ of χ it follows that g acts via the same scalar multiple of identity on each simple constituent of $M_\chi\downarrow_{\bar{G}(H)}$. Then all these simple constituents are isomorphic and $\chi\downarrow_{k\bar{G}(H)} = \chi(1)\psi$. Since $\chi_{\mathrm{ad}}\downarrow_{k\bar{G}(H)} = \chi_{\mathrm{ad}}(1)\epsilon$ we get that $\chi \approx \mu$ if and only if

$$\frac{1}{\chi(1)}\chi\downarrow_{\bar{G}(H)} = \frac{1}{\mu(1)}\mu\downarrow_{\bar{G}(H)}.$$

(3) It follows from (2) immediately. □

For any irreducible character χ let $G_\chi = \sum_{\chi_i \approx \chi} \chi_i(1)\chi_i$. If an element $g \in \bar{G}(H)$ acts as a scalar on a module M of H then it acts as the same scalar on each simple submodule of M. In particular, all the irreducible constituents of $\chi\mu$ are in the same equivalence class of $\approx$. Using this we denote by $G_{\chi\mu}$ the element G_η for some irreducible constituent η of $\chi\mu$.

Proposition 25. *Assume H is a semisimple Hopf algebra with $C(H)$ commutative. If χ and μ are two irreducible characters of H then the following relations hold*:

(1) *$\chi G_\mu = \chi(1)G_{\chi\mu}$ and $G_\chi(1) = \frac{\dim_k H}{|\bar{G}(H)|}$ for every irreducible character χ.*
(2) *$G_\chi G_\mu = \frac{n}{|\bar{G}(H)|}G_{\chi\mu}$.*
(3) *G_χ is a central element of H^* for any irreducible character $\chi \in C(H)$.*

Proof. (1) If t denotes the regular character of H then $\chi t = \chi(1)t$. On the other hand,

$$t = \sum_{i=0}^{r} \chi_i(1)\chi_i = \sum_{\chi/\approx} G_\chi,$$

where in the last sum χ runs over all the representatives of the equivalence classes of $\approx$. The remark above implies that $\chi G_\mu = \chi(1)G_{\chi\mu}$. In particular, for $\mu = \chi_0$, the trivial character of H we get that $\chi G_{\chi_0} = \chi(1)G_\chi$. Applying 1 to both sides of the last equality it follows that $G_{\chi_0}(1) = G_\chi(1)$ for every irreducible character χ. Then the above formula for t implies that

$$G_\chi(1) = \frac{n}{|G|}$$

for every irreducible character χ.

(2) First let us observe that if $\chi_i \approx \chi$ then $G_{\chi_i\mu} = G_{\chi\mu}$ since all the central grouplike elements $g \in \bar{G}(H)$ act via the same scalar on both $\chi_i\mu$ and $\chi\mu$. Thus

$$G_\chi G_\mu = \sum_{\chi_i \approx \chi} \chi_i(1)\chi_i G_\mu = \sum_{\chi_i \approx \chi} \chi_i(1)^2 G_{\chi\mu} = G_\chi(1)G_{\chi\mu} = \frac{n}{|\bar{G}(H)|} G_{\chi\mu}.$$

(3) For every central grouplike element $g \in \bar{G}(H)$ we have

$$F_g = \sum_i \chi_{i^*}(g)\chi_i = \sum_{\chi/\approx} \frac{\chi(g)}{\chi(1)} G_\chi,$$

where the last sum is over all the representatives of the equivalence classes of $\approx$. The matrix $(\frac{\chi(g)}{\chi(1)})_{\chi,g}$ is nondegenerate which implies that every G_χ is a linear combination of the elements $(F_g)_{g\in\bar{G}(H)}$ and therefore central by Lemma 11. One can write

$$G_\chi = \frac{n}{|\bar{G}(H)|} \sum_{g\in\bar{G}(H)} \frac{\chi(g)}{\chi(1)} \xi_g. \qquad \square \tag{4.1}$$

Corollary 26. *Let $H = kG$ for a finite group G. Then $\sim$ is an equivalence relation if and only if* $\mathrm{Ann}(H_{\mathrm{ad}}) = \omega(k\mathcal{Z}(G))kG$.

Example 1 [18]. Let p and q be two prime numbers with $q-1$ divisible by p. We will construct a group G such that $\mathcal{Z}(G) = 1$ but $\mathrm{Ann}((kG)_{\mathrm{ad}}) \neq 0$. Let P be an elementary abelian p-group of order p^2 and Q be an elementary abelian q-group of order q^{p+1}. Then $G = Q \rtimes P$ where the action of P on Q is constructed such that each subgroup of P of order p is the kernel of the action of P on a cyclic factor of Q. Suppose $Q = Q_0 \times Q_1 \times \cdots \times Q_p$ where each Q_i is cyclic of order q. If $P_0, P_1, \ldots, P_p$ are all the subgroups of P of order p then we define the action of P such that each P_i acts trivially on Q_i. This is possible since $p \mid q-1$. It follows that for each $g \in G$ there is i such that $C_G(g) \supseteq Q_i$. Therefore $\omega(C_G(g)) \supseteq \prod_{i=0}^{p} \omega(Q_i) \neq 0$. In the same paper [18] it is shown that $\mathrm{Ann}(H_{\mathrm{ad}}) = \bigcap_{g\in G} \omega(C_G(g))$. Therefore $\mathrm{Ann}(H_{\mathrm{ad}}) \neq 0$ although $\mathcal{Z}(G) = 1$. The previous corollary implies that $\sim$ is not transitive.

In the same paper [18] it was proved that the adjoint action on S_n is faithful, therefore $\sim$ is an equivalence relation and in this case there is only one equivalence class.

Lemma 27. *Let H be a semisimple Hopf algebra with $C(H)$ commutative and ψ be an irreducible character of $\bar{G}(H)$. Then $\psi\uparrow^{H}_{k\bar{G}(H)} = G_\chi$ for some irreducible character $\chi \in C(H)$.*

Proof. Recall that $G_\chi = \sum_{\chi_i \approx \chi} \chi_i(1)\chi_i$ and $\chi \approx \mu$ if and only if

$$\frac{1}{\chi(1)}\chi\downarrow_{\bar{G}(H)} = \frac{1}{\mu(1)}\mu\downarrow_{\bar{G}(H)}.$$

The relation follows from Frobenius reciprocity for $\bar{G}(H)$ and H. $\square$

Lemma 28. *Let H be a semisimple Hopf algebra with $C(H)$ commutative. If $D(H)_{\mathrm{ad}}$ is the adjoint representation of $D(H)$ then $D(H)_{\mathrm{ad}}\downarrow_H \cong H_{\mathrm{ad}}^{\otimes 2}$.*

Proof. Since $C(H)$ is commutative, with the notations from relation (1.1) we have $p_s = 1$ for every $0 \leqslant s \leqslant f$ and

$$\chi_{\mathrm{ad}} = \sum_{s=0}^{f} \frac{1}{a_s} e_s$$

where χ_{ad} is the character of the adjoint representation H_{ad}. Similarly,

$$D(H)_{\mathrm{ad}} = \sum_{s=0}^{l} \frac{1}{A_s} E_s.$$

Then

$$D(H)_{\mathrm{ad}}\downarrow_H = \sum_{s=0}^{f} \frac{1}{A_s} e_s = \chi_{\mathrm{ad}}^2$$

since $A_s = a_s^2$. $\square$

Theorem 29. *Let H be a semisimple Hopf algebra with $C(H)$ commutative. Let μ be an irreducible character of $D(H)$ and $\mathcal{D}_\mu$ be the equivalence class of μ. If χ is an irreducible constituent of $\mu\downarrow_H$ then $\mathcal{D}_\mu\downarrow_H = \frac{n}{l} G_\chi$ where l is the index of $\bar{G}(H)$ inside $\bar{G}(D(H))$.*

Proof. $\mathcal{D}_\mu$ is a central character in $D(H)^*$ and by Proposition 25

$$\mathcal{D}_\mu = \frac{n^2}{|\bar{G}(D(H))|} \sum_{x \in \bar{G}(D(H))} \frac{\mu(x)}{\mu(1)} \xi_x.$$

Let $\psi : D(H)^* \to D(H)$ be the map defined in Section 2 which shows that $D(H)$ is a factorizable Hopf algebra. Since every central grouplike element of $x \in \bar{G}(D(H))$ is of the type $x = f \bowtie g$ for some $f \in G(H^*)$ and $g \in G(H)$, it follows that $\psi(\xi_x)$ is the central idempotent of H corresponding to the simple one-dimensional $D(H)$-module $V_{g,f}$. Therefore $\xi_x\downarrow_H = \xi_g$ if g is a central grouplike element of H and $f = 1$ and $\xi_x\downarrow_H = 0$ otherwise. Consequently,

$$\mathcal{D}_\mu\downarrow_H = \frac{n^2}{|\bar{G}(D(H))|}\sum_{g\in\bar{G}(H)}\frac{\mu(\epsilon\bowtie g)}{\mu(1)}\xi_g.$$

On the other hand,

$$\frac{1}{\mu(1)}\mu\downarrow_{\bar{G}(H)} = \frac{1}{\chi(1)}\chi\downarrow_{\bar{G}(H)}$$

since the irreducible constituents of $\mu\downarrow_H$ are equivalent with χ. Then

$$\frac{\mu(\epsilon\bowtie g)}{\mu(1)} = \frac{\chi(g)}{\chi(1)}$$

for any $g \in \bar{G}(H)$. It follows that

$$\mathcal{D}_{\mu\downarrow_H} = \frac{n^2}{|\bar{G}(D(H))|}\sum_{g\in\bar{G}(H)}\frac{\chi(g)}{\chi(1)}\xi_g = \frac{n|\bar{G}(H)|}{|\bar{G}(D(H))|}G_\chi = \frac{n}{l}G_\chi. \qquad \square$$

5. The Drinfel'd double of the eight-dimensional Hopf algebra

In this section we describe the Grothendieck ring structure of the Drinfel'd double of the unique nontrivial eight-dimensional Hopf algebra H_8 [6,10].

H_8 can be presented by generators x, y, z with relations

$$x^2 = y^2 = 1,$$
$$xy = yx, \qquad zx = yz, \qquad zy = xz,$$
$$2z^2 = (1 + x + y - xy).$$

The coalgebra structure is determined by

$$\Delta(x) = x\otimes x, \qquad \epsilon(x) = 1, \qquad S(x) = x,$$
$$\Delta(y) = y\otimes y, \qquad \epsilon(y) = 1, \qquad S(y) = y,$$
$$\Delta(z) = \frac{1}{2}\big((1+y)\otimes 1 + (1-y)\otimes x\big)(z\otimes z),$$
$$\epsilon(z) = 1, \qquad S(z) = z^{-1}.$$

In addition $H_8 \simeq H_8^*$, $G(H_8) = \{1, x, y, xy\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, and $\bar{G}(H_8) = G(H_8) \cap Z(H_8) = \{1, xy\} \simeq \mathbb{Z}_2$.

As an algebra, $H_8 \cong k^4 \times M_2(k)$. It follows that H_8 has five irreducible characters, four one-dimensional $\epsilon, u_1, u_2, u_1u_2$, and one two-dimensional self dual character χ. Therefore, the character ring of H_8 is five-dimensional and the ring structure is given by $G(H_8^*) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\chi^2 = \epsilon + u_1 + u_2 + u_1u_2$. The relation $\approx$ on the simple H_8-modules has 2 equivalence classes given by $G_0 = \epsilon + u_1 + u_2 + u_1u_2$ and $G_1 = 2\chi$. Since H_8 is self dual Hopf algebra H_8^* has the same representation type as H_8. Let $1, \hat{u}_1, \hat{u}_2, \hat{u}_1\hat{u}_2$ be the four one-dimensional representations of H_8^* and $\hat{\chi}$ be the two-dimensional representation of H_8^*. Similarly H_8^* has two equivalence classes, given by $\hat{G}_0 = 1 + \hat{u}_1 + \hat{u}_2 + \hat{u}_1\hat{u}_2$ and $\hat{G}_1 = 2\hat{\chi}$.

In [13] it was proved that $D(H_8) \cong k^8 \times M_2(k)^{14}$ and $D(H_8)^* \cong k^{16} \times M_2(k)^8 \times M_4(k)$ as algebras.

Remark 30. Let A be a finite-dimensional Hopf algebra and let $g \in G(A)$, $\eta \in G(A^*)$. Let $V_{g,\eta}$ denote the vector space $k1$ endowed with the action $h.1 = \eta(h)1$, $h \in H$, and the coaction $1 \mapsto g \otimes 1$. By [20], the one-dimensional $D(A)$-modules over A are exactly of the form $V_{g,\eta}$, where $g \in G(A)$ and $\eta \in G(A^*)$ are such that $(\eta \rightharpoonup h)g = g(h \leftharpoonup \eta)$, for all $h \in A$. In particular, if $g \in Z(A)$ and $\eta \in Z(A^*)$ then $V_{g,\epsilon}$ and $V_{1,\eta}$ are one-dimensional $D(A)$-modules and $\epsilon \otimes g, 1 \otimes \eta \in G(D(A)^*)$.

Let $g \in G(H_8) \backslash Z(H_8)$ and $\eta \in G(H_8^*) \backslash Z(H_8^*)$. Then according to [13, Lemma 15.2.1] $V_{g,\eta}$ is a $D(H_8)$-module and $G(D(H_8)^*) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. In order to determine the Grothendieck ring structure of $D(H_8)$ we need to determine first the equivalence classes under $\approx$. Since $|\bar{G}(D(H_8))| = 8$ there are eight equivalence classes and the dimension of the representative character of each equivalence class is 8. Therefore each equivalence class contains either two two-dimensional representations, one two-dimensional representation and four one-dimensional representation, or eight one-dimensional representation. Since both $C(H_8)$ and $C(H_8^*)$ are commutative it follows that

$$D_{\mathrm{ad}} \downarrow_{H_8} = \chi_{\mathrm{ad}}^2 = 5\epsilon + u_1 + u_2 + u_1u_2.$$

Similarly

$$D_{\mathrm{ad}} \downarrow_{H_8^*} = 5 \cdot 1 + \hat{u}_1 + \hat{u}_2 + \hat{u}_1\hat{u}_2.$$

The equivalence class of the trivial $D(H_8)$-module $V_{1,\epsilon}$ is denoted by $\mathcal{D}_0$ and has the restriction G_0 to H and $\hat{G}_0$ to H^*. The restrictions of $D(H_8)$-modules to H_8 and H_8^* can be described using Proposition 4. Looking in Table 1 it follows that $\mathcal{D}_0$ might contain any of the one-dimensional representations and possibly V_9 or V_{10}. Since $\mathcal{D}_0$ cannot contain both of these two-dimensional modules, the self duality of H_8 implies that this class contains all the eight one-dimensional representations. Therefore all the other equivalence classes have 2 representations of dimension two. Comparing the restrictions of these modules to both H_8 and H_8^* and using Theorem 29 it follows immediately that $\{V_1, V_3\}$, $\{V_2, V_4\}$, $\{V_5, V_7\}$, $\{V_6, V_8\}$ $\{V_9, V_{10}\}$ form equivalence classes. Without loss of generality

Table 1
$D(H_8)$-modules and their restrictions

$D(H_8)$-modules	Restriction to H_8	Restriction to H_8^*
$V_{1,\epsilon}$	ϵ	1
V_{1,u_1u_2}	u_1u_2	1
$V_{\hat{u}_1\hat{u}_2,\epsilon}$	ϵ	$\hat{u}_1\hat{u}_2$
$V_{\hat{u}_1,u_1}$	u_1	$\hat{u}_1$
$V_{\hat{u}_1,u_2}$	u_2	$\hat{u}_1$
$V_{\hat{u}_2,u_1}$	u_1	$\hat{u}_2$
$V_{\hat{u}_2,u_2}$	u_2	$\hat{u}_2$
$V_{\hat{u}_1\hat{u}_2,u_1u_2}$	u_1u_2	$\hat{u}_1\hat{u}_2$
V_1	$\epsilon + u_1$	$\hat{\chi}$
V_2	$\epsilon + u_2$	$\hat{\chi}$
$V_3 = V_{1,u_1u_2} \otimes V_1$	$u_1u_2 + u_1$	$\hat{\chi}$
$V_4 = V_{1,u_1u_2} \otimes V_2$	$u_1u_2 + u_2$	$\hat{\chi}$
V_5	χ	$1 + \hat{u}_1$
V_6	χ	$1 + \hat{u}_2$
$V_7 = V_{\hat{u}_1\hat{u}_2,\epsilon} \otimes V_5$	χ	$\hat{u}_1\hat{u}_2 + \hat{u}_2$
$V_8 = V_{\hat{u}_1\hat{u}_2,\epsilon} \otimes V_6$	χ	$\hat{u}_1\hat{u}_2 + \hat{u}_1$
V_9	$\epsilon + u_1u_2$	$\hat{u}_1 + \hat{u}_2$
V_{10}	$u_1 + u_2$	$1 + \hat{u}_1\hat{u}_2$
$V_{11}, V_{12}, V_{13}, V_{14}$	χ	$\hat{\chi}$

it might be assumed that $\{V_{11}, V_{12}\}$ and $\{V_{13}, V_{14}\}$ are the other two equivalence classes. Let $\mathcal{D}_1 = \{V_9, V_{10}\}$, $\mathcal{D}_2 = \{V_5, V_7\}$ and $\mathcal{D}_3 = \{V_1, V_8\}$. Proposition 25, part (2) implies that any equivalence class is obtained as a product from other equivalence classes. Since $\bar{G}(D(H_8)) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ has three group generators it follows that all the equivalence classes can be obtained as a product from three different equivalence classes. We claim that $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ generate all the other equivalence classes. Indeed, the restrictions of these classes to both H_8 and H_8^* give that $\mathcal{D}_4 = \{V_6, V_8\} = \mathcal{D}_1\mathcal{D}_2$, $\mathcal{D}_5 = \{V_2, V_4\} = \mathcal{D}_1\mathcal{D}_3$, $\mathcal{D}_6 = \{V_{11}, V_{12}\} = \mathcal{D}_2\mathcal{D}_3$ and $\mathcal{D}_7 = \{V_{13}, V_{14}\} = \mathcal{D}_1\mathcal{D}_2\mathcal{D}_3$. Examining Table 1 it follows that multiplying two $D(H_8)$-modules the result cannot have a constituent with multiplicity 2 since its restriction to either H_8 or H_8^* does not have this property. Therefore, the multiplication of two modules from two different equivalence classes should be the sum of the two modules in the corresponding product class. In this way the multiplication of any 2 two-dimensional modules can be determined if they are from two different equivalence classes. If they are in the same equivalence class, their product is the sum of 4 one-dimensional modules that can be easily determined just looking at the restrictions of the product to both H_8 and H_8^*.

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